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TR-EE64-14
AFCRL-64-658

PURDUE UNIVERSITY
SCHOOL OF ELECTRICAL ENGINEERING

The Theoretical and Numerical Determination of the Radar Cross Section of a Finite Cone

F. V. Schultz, G. M. Ruckgaber, S. Richter, and J. K. Schindler
Purdue Research Foundation
Lafayette, Indiana

Contract No. AF 19(628)-1691
Project No. 5635
Task No. 563502
Scientific Report No. 1
August, 1964

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PREPARED FOR
AIR FORCE CAMBRIDGE RESEARCH LABORATORIES
OFFICE OF AEROSPACE RESEARCH
UNITED STATES AIR FORCE
BEDFORD, MASSACHUSETTS

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ABSTRACT

This investigation of the radar cross-section of a finite cone can be divided into three areas. First, the exact solution for the scattering of a plane electromagnetic wave by a finite cone is presented. Rigorous electromagnetic theory is used in the solution, and no approximations are made. Secondly, methods of obtaining numerical results for the radar cross-section from the analytic solution by using a digital computer are discussed. The third area is a presentation and discussion of the numerical results obtained.

LIST OF CONTRIBUTORS

Schultz, F. V., Project Supervisor

Kaul, R. K.

Richter, S.

Ruckgaber, G. M.

Schindler, J. K.

RELATED CONTRACTS AND PUBLICATIONS

The present contract is a continuation of Contract No. 19(604)4051.

On this earlier contract the following publications were produced:

Rogers, C. C. and F. V. Schultz, "The Scattering of a Plane Electromagnetic Wave by a Finite Cone", School of Electrical Engineering, Purdue University, Report No. ERD-TN-60-765, August, 1960.

Rogers, C. C., J. K. Schindler, and F. V. Schultz, "The Scattering of a Plane Electromagnetic Wave by a Finite Cone", presented at URSI Symposium on Electromagnetic Theory and Antennas, Copenhagen, Denmark, June, 1962. Also published in Symposium Proceedings, Pergamon Press, 1963.

Schultz, F. V., D. M. Bolle, and J. K. Schindler, "The Scattering of Electromagnetic Waves by Perfectly Reflecting Objects of Complex Shape", School of Electrical Engineering, Purdue University, Report No. AFCRL-63-319 (Final Report), January, 1963.

Schindler, J. K and F. V. Schultz, "The Determination of the Electromagnetic Scattering from a Cavity Backed Plane Surface" in preparation as an AFCRL Research Report.

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THE THEORETICAL AND NUMERICAL DETERMINATION
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1. Statement of the Problem

The problem undertaken is the exact solution for the scattering of a plane electromagnetic wave by a finite perfectly conducting cone. We consider only "nose-on" incidence (Figure 1). In order that the entire surface of the cone can be expressed as a constant co-ordinate surface in spherical co-ordinates, the end cap of the cone is taken to be a segment of a spherical surface with center at the apex of the cone. Time variations are assumed to be given by $e^{i\omega t}$ and MKS units are used.

A considerable amount of effort, both theoretical and experimental, has been devoted to the cone scattering problem by many workers. No attempt is made here to summarize the work, in view of the very excellent summary which appears in the report by Kleinman and Senior (1963). It should be noted that the present work is an extension of that done earlier by Rogers and Schultz (1960), and by Rogers, Schindler, and Schultz (1962).

This scattering problem is treated herein as a boundary-value problem in electromagnetic theory and no physical approximations are made. The partial differential equation is, of course, the vector Helmholtz equation,

$$\nabla^2 \vec{C} + k^2 \vec{C} = 0 \quad , \tag{1}$$

where $k = 2\pi/\lambda$ and $\vec{C}$ may be either the electric field vector $\vec{E}$ or

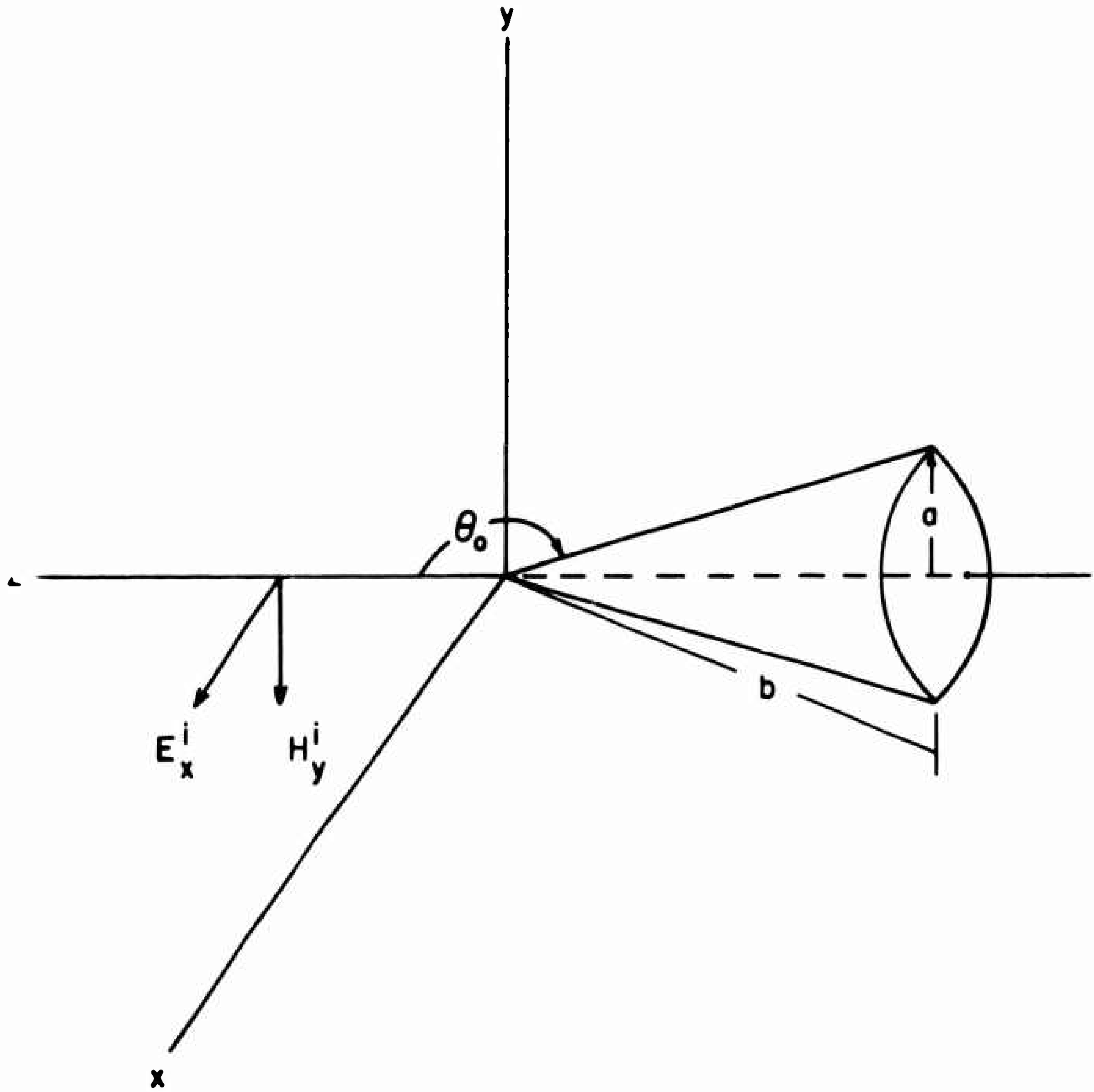

Fig. 1. Cone Configuration

the magnetic field vector $\vec{H}$. Solutions of (1) are obtained in the form of infinite series containing unknown constants. To complete the solution of the problem, these constants are determined by satisfying the necessary boundary conditions for $\vec{E}$ and $\vec{H}$ on the surface of the perfectly conducting cone, the radiation condition at infinity, and the finite energy condition.

Numerical results have been obtained, and these are compared with experimental results obtained elsewhere, as well as with theoretical results obtained with the use of approximate methods.

2. Solution of Vector Helmholtz Equation

The procedure used here for obtaining the solutions of the vector Helmholtz equation is well known (Stratton, 1941).

Solutions of (1) are

$$\vec{\ell} = \nabla \Phi ,$$

$$\vec{m} = \nabla \times (\Phi \vec{r}) , \tag{2}$$

$$\vec{n} = \frac{1}{k} \nabla \times \vec{m} ,$$

where $\vec{r}$ is the radial vector in spherical co-ordinates and Φ is the solution of the scalar Helmholtz equation

$$\nabla^2 \Phi + k^2 \Phi = 0 . \tag{3}$$

In the region surrounding the cone, $\nabla \cdot \vec{E} = \nabla \cdot \vec{H} = 0$. Since $\nabla \cdot \vec{\ell} \neq 0$, we use only the $\vec{m}$ and $\vec{n}$ solutions to represent $\vec{E}$ and $\vec{H}$.

It is also well known that the solution of (3) is

$$\Phi^{n}_{\substack{e \\ o}mv}(r,\theta,\phi) = z^{n}_{v}(kr)\, P^{m}_{v}(\cos\theta) \begin{bmatrix} \cos m\phi \\ \sin m\phi \end{bmatrix}, \qquad (4)$$

where n can have the values 1, 2, 3, or 4 to represent Bessel functions of the first kind ($j_n(kr)$), Bessel functions of the second kind ($n_n(kr)$), Hankel functions of the first kind ($h^1_n(kr)$), and Hankel functions of the second kind ($h^2_n(kr)$), respectively. $P^m_v(\cos\theta)$ is an associated Legendre function of degree v and order m, and we let e signify "even" and o signify "odd" for $\cos m\phi$ and $\sin m\phi$, respectively.

The desired solutions of the vector Helmholtz equation are then obtained from (2) and (4):

$$\vec{m}^{n}_{\substack{e \\ o}mv} = \mp \frac{m}{\sin\theta}\, z^{n}_{v}(kr)\, P^{m}_{v}(\cos\theta) \begin{bmatrix} \sin m\phi \\ \cos m\phi \end{bmatrix} \vec{a}_{\theta} - z^{n}_{v}(kr)\, \frac{dP^{m}_{v}}{d\theta} \begin{bmatrix} \cos m\phi \\ \sin m\phi \end{bmatrix} \vec{a}_{\phi} \qquad (5)$$

$$\vec{n}^{n}_{\substack{e \\ o}mv} = \frac{v(v+1)}{kr}\, z^{n}_{v}(kr)\, P^{m}_{v}(\cos\theta) \begin{bmatrix} \cos m\phi \\ \sin m\phi \end{bmatrix} \vec{a}_{r} + z^{n\,\prime}_{v}(kr)\, \frac{dP^{m}_{v}}{d\theta} \begin{bmatrix} \cos m\phi \\ \sin m\phi \end{bmatrix} \vec{a}_{\theta} \mp \frac{m}{\sin\theta}\, z^{n\,\prime}_{v}(kr)\, P^{m}_{v}(\cos\theta) \begin{bmatrix} \sin m\phi \\ \cos m\phi \end{bmatrix} \vec{a}_{\phi} \qquad (6)$$

where $z^{n\,\prime}_{v}(kr) = \frac{1}{kr}\frac{d}{dr}\left[r\, z^{n}_{v}(kr) \right]$, and $\vec{a}_r$, $\vec{a}_\theta$, and $\vec{a}_\phi$ are the spherical unit vectors.

3. Space Sectionalization

One of the most important characteristics of the solution of this problem is that of dividing the space surrounding the cone into two regions to facilitate the field expansions and application of the boundary conditions. The $\vec{E}$ and $\vec{H}$ fields are then expanded in terms of the radial and spherical functions appropriate to each region.

Since the scattered fields must be spherically diverging waves for large values of the co-ordinate r, the use of Hankel functions is obvious since they possess the desired wave behavior as $r \to \infty$. In particular, since we assume a time variation of the form $e^{i\omega t}$, the use of $z_n^4(kr) = h_n^2(kr)$ functions is necessary to achieve an outward traveling wave. At the tip of the cone, however, the Hankel functions possess a singularity the order of which is too large to satisfy the finite energy condition. This characteristic of the radial functions suggests a division of the two regions at a finite value of r.

The behavior of the associated Legendre functions indicates a division of the two regions at $r = b$. This is then the surface that we use to separate regions I and II (Figure 2). In region II, the fields exist and are bounded everywhere in the complete θ domain of $\theta = 0$ to π, requiring the use of only associated Legendre functions of integral degree. In region I, however, $\theta = \pi$ is not in the domain of interest, allowing the use of associated Legendre functions of non-integral degree. It will be seen that the boundary conditions will determine the non-integral degree of each associated Legendre function to be used in region I.

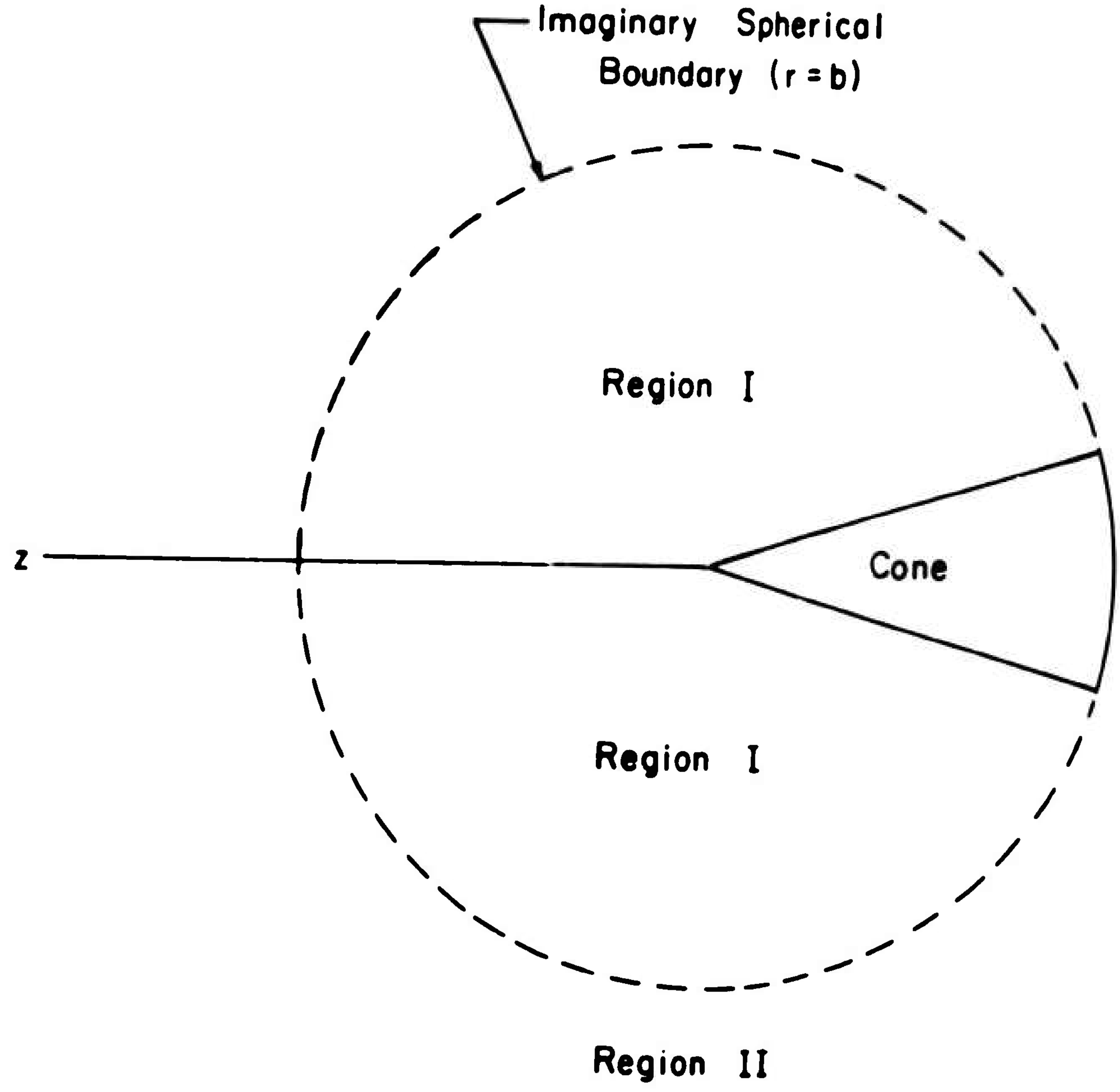

Fig. 2. Space Sectionalization

The reader may wish to refer to Rogers and Schultz (1960) for a more complete discussion of the selection of the various space divisions possible.

4. Field Expansions

In region I the total fields are designated by $\vec{E}_I^t$ and $\vec{H}_I^t$. In region II we wish to keep the incident and scattered fields separate, and designate the incident fields by $\vec{E}_{II}^i$ and $\vec{H}_{II}^i$ and the scattered fields by $\vec{E}_{II}^s$ and $\vec{H}_{II}^s$.

In region II the incident electric field may be expressed (Stratton, 1941) as

$$\vec{E}_{II}^i = e^{ikz}\,\vec{a}_x = e^{ikr\cos\theta}\,\vec{a}_x = \sum_{n=1}^{\infty}\left(\gamma_n\,\vec{m}_{oln}^1 + \Gamma_n\,\vec{n}_{eln}^1\right), \tag{7}$$

where

$$\gamma_n = i^n\,\frac{2n+1}{n(n+1)}\,, \qquad \Gamma_n = -i^{n+1}\,\frac{2n+1}{n(n+1)}\,, \tag{8}$$

and $\vec{a}_x$ is a unit vector in the x direction. The ϕ variation in the incident field requires that $m=1$ and forces us to use odd $\vec{m}$ functions and even $\vec{n}$ functions in all expansions of the electric field.

The scattered field in region II is written as

$$\vec{E}_{II}^s = \sum_{n=1}^{\infty}\left(c_n\,\vec{m}_{oln}^4 + d_n\,\vec{n}_{eln}^4\right), \tag{9}$$

where c_n and d_n are expansion coefficients to be determined from the boundary conditions. Here we have selected $z_n^4(kr) = h_n^2(kr)$ and the $\vec{m}$ and $\vec{n}$ functions as previously discussed.

In region I the total electric field is expressed as

$$\vec{E}_I^t = \sum_\nu a_\nu \vec{m}^1_{o1\nu} + \sum_\mu b_\mu \vec{n}^1_{e1\mu} . \tag{10}$$

Here a_ν and b_μ are expansion coefficients to be determined, and μ and ν are the non-integral degrees of the associated Legendre functions, which are also yet to be determined.

The analogous representations for the magnetic field are obtained from Maxwell's equations,

$$\nabla \times \vec{E} = -i\,\omega\,\mu_o\,\vec{H} \quad , \qquad \nabla \times \vec{H} = i\,\omega\,\epsilon_o\,\vec{E} \quad , \tag{11}$$

and the relations,

$$\nabla \times \vec{m} = k\,\vec{n} \quad , \qquad \nabla \times \vec{n} = k\,\vec{m} \quad . \tag{12}$$

By using (7) through (10), in addition to (11) and (12), and noting that $k = \omega\sqrt{\mu_o \epsilon_o}$, one obtains the expressions for the magnetic fields:

$$\vec{H}^1_{II} = \frac{i}{\eta}\left[\sum_{n=1}^{\infty}\left(\gamma_n \vec{n}^1_{o1n} + \Gamma_n \vec{m}_{e1n}\right)\right] , \tag{13}$$

$$\vec{H}^s_{II} = \frac{i}{\eta}\left[\sum_{n=1}^{\infty}\left(c_n \vec{n}^4_{o1n} + d_n \vec{m}^4_{e1n}\right)\right] , \tag{14}$$

$$\vec{H}^t_I = \frac{i}{\eta}\left[\sum_\nu a_\nu \vec{n}^1_{o1\nu} + \sum_\mu b_\mu m^1_{e1\mu}\right] , \tag{15}$$

where η is the intrinsic impedance of free space, $\sqrt{\mu_o/\epsilon_o}$.

For future reference, the field quantities are now expanded in their entirety:

$$\vec{E}_I^t = \left[\sum_\mu b_\mu \frac{\mu(\mu+1)}{kr} j_\mu(kr) P_\mu^1(\cos\theta) \right] \cos\phi\, \vec{a}_r$$

$$+ \left[\sum_\upsilon a_\upsilon j_\upsilon(kr) \frac{P_\upsilon^1(\cos\theta)}{\sin\theta} + \sum_\mu b_\mu j_\mu'(kr) \frac{dP_\mu^1}{d\theta} \right] \cos\phi\, \vec{a}_\theta$$

$$- \left[\sum_\upsilon a_\upsilon j_\upsilon(kr) \frac{dP_\upsilon^1}{d\theta} + \sum_\mu b_\mu j_\mu'(kr) \frac{P_\mu^1(\cos\theta)}{\sin\theta} \right] \sin\phi\, \vec{a}_\phi \; . \tag{16}$$

$$\vec{E}_{II}^i = \sum_{n=1}^{\infty} \left\{ \left[\Gamma_n \frac{n(n+1)}{kr} j_n(kr) P_n^1(\cos\theta) \right] \cos\phi\, \vec{a}_r \right.$$

$$+ \left[\gamma_n j_n(kr) \frac{P_n^1(\cos\theta)}{\sin\theta} + \Gamma_n j_n'(kr) \frac{dP_n^1}{d\theta} \right] \cos\phi\, \vec{a}_\theta$$

$$\left. - \left[\gamma_n j_n(kr) \frac{dP_n^1}{d\theta} + \Gamma_n j_n'(kr) \frac{P_n^1(\cos\theta)}{\sin\theta} \right] \sin\phi\, \vec{a}_\phi \right\} . \tag{17}$$

$$\vec{E}_{II}^s = \sum_{n=1}^{\infty} \left\{ \left[d_n \frac{n(n+1)}{kr} h_n(kr) P_n^1(\cos\theta) \right] \cos\phi\, \vec{a}_r \right.$$

$$+ \left[c_n h_n(kr) \frac{P_n^1(\cos\theta)}{\sin\theta} + d_n h_n'(kr) \frac{dP_n^1}{d\theta} \right] \cos\phi\, \vec{a}_\theta$$

$$\left. - \left[c_n h_n(kr) \frac{dP_n^1}{d\theta} + d_n h_n'(kr) \frac{P_n^1(\cos\theta)}{\sin\theta} \right] \sin\phi\, \vec{a}_\phi \right\} . \tag{18}$$

$$\vec{H}_I^t = \frac{i}{\eta} \left\{ \left[\sum_\upsilon a_\upsilon \frac{\upsilon(\upsilon+1)}{kr} j_\upsilon(kr) P_\upsilon^1(\cos\theta) \right] \sin\phi\, \vec{a}_r \right.$$

$$+ \left[\sum_\upsilon a_\upsilon j_\upsilon'(kr) \frac{dP_\upsilon^1}{d\theta} - \sum_\mu b_\mu j_\mu(kr) \frac{P_\mu^1(\cos\theta)}{\sin\theta} \right] \sin\phi\, \vec{a}_\theta$$

$$\left. + \left[\sum_\upsilon a_\upsilon j_\upsilon'(kr) \frac{P_\upsilon^1(\cos\theta)}{\sin\theta} - \sum_\mu b_\mu j_\mu(kr) \frac{dP_\mu^1}{d\theta} \right] \cos\phi\, \vec{a}_\phi \right\} \tag{19}$$

$$\vec{H}^{i}_{II} = \frac{1}{\eta} \sum_{n=1}^{\infty} \left\{ \left[\gamma_n \frac{n(n+1)}{kr} j_n(kr) P^1_n(\cos\theta) \right] \sin\phi \, \vec{a}_r \right.$$

$$+ \left[\gamma_n j'_n(kr) \frac{dP^1_n}{d\theta} - \Gamma_n j_n(kr) \frac{P^1_n(\cos\theta)}{\sin\theta} \right] \sin\phi \, \vec{a}_\theta$$

$$\left. + \left[\gamma_n j'_n(kr) \frac{P^1_n(\cos\theta)}{\sin\theta} - \Gamma_n j_n(kr) \frac{dP^1_n}{d\theta} \right] \cos\phi \, \vec{a}_\phi \right\} . \tag{20}$$

$$\vec{H}^{s}_{II} = \frac{1}{\eta} \sum_{n=1}^{\infty} \left\{ \left[c_n \frac{n(n+1)}{kr} h_n(kr) P^1_n(\cos\theta) \right] \sin\phi \, \vec{a}_r \right.$$

$$+ \left[c_n h'_n(kr) \frac{dP^1_n}{d\theta} - d_n h_n(kr) \frac{P^1_n(\cos\theta)}{\sin\theta} \right] \sin\phi \, \vec{a}_\theta$$

$$\left. + \left[c_n h'_n(kr) \frac{P^1_n(\cos\theta)}{\sin\theta} - d_n h_n(kr) \frac{dP^1_n}{d\theta} \right] \cos\phi \, \vec{a}_\phi \right\} . \tag{21}$$

Equations (16) through (21) contain six sets of unknown constants, μ, ν, a_ν, b_μ, c_n, and d_n. These are to be determined by satisfying the boundary conditions.

5. Boundary Conditions

We have already satisfied the finite energy condition at the tip of the cone and the radiation condition at infinity by the proper choice of radial functions in each region. The following boundary conditions remain to be satisfied:

$$\text{(a)} \quad \left[\vec{E}^{t}_{I} \right]_{r,\phi} = 0 \quad \text{for} \quad \theta = \theta_o , \quad r \leq b \tag{22a}$$

$$\text{(b)} \quad \left[\vec{E}^{i}_{II} + \vec{E}^{s}_{II} \right]_{\theta,\phi} = \left\{ \begin{array}{ll} \left[\vec{E}^{t}_{I} \right]_{\theta,\phi} & \text{for } 0 \leq \theta < \theta_o \\ 0 & \text{for } \theta_o \leq \theta \leq \pi \end{array} \right\} r = b \tag{22b}$$

(c) $\left[\vec{H}^{i}_{II} + \vec{H}^{s}_{II} \right]_{\theta,\phi} = \left[\vec{H}^{t}_{I} \right]_{\theta,\phi}$ for $r = b, \ 0 \le \theta < \theta_o$ (22c)

(d) The finite energy condition at the edge of the cone

$(r \to b, \ \theta \to \theta_o)$, (22d)

where b is the radius of the spherical cap and θ_o is half of the exterior apex angle.

6. The Solution

To satisfy boundary condition (22a) we first equate the r-component of $\vec{E}^{t}_{I}$ to zero at $\theta = \theta_o$,

$$\sum_{\mu} b_{\mu} \frac{\mu(\mu + 1)}{kr} j_{\mu}(kr) P^{1}_{\mu}(\cos \theta_o) \cos \phi = 0 , \tag{23}$$

and thus set

$$P^{1}_{\mu}(\cos \theta_o) = 0 . \tag{24}$$

This equation determines the values of μ. Equating the ϕ-component of $\vec{E}^{t}_{I}$ to zero at $\theta = \theta_o$ gives

$$\sum_{\nu} a_{\nu} j_{\nu}(kr) \left. \frac{dP^{1}_{\nu}}{d\theta} \right|_{\theta=\theta_o} + \sum_{\mu} b_{\mu} j'_{\mu}(kr) \frac{P^{1}_{\mu}(\cos \theta_o)}{\sin \theta_o} = 0 . \tag{25}$$

Since $P^{1}_{\mu}(\cos \theta_o) = 0$ by (24), we set

$$\left. \frac{dP^{1}_{\nu}}{d\theta} \right|_{\theta=\theta_o} = 0 \tag{26}$$

and thus the values of ν are determined.

Next, the boundary conditions (22b) and (22c) are applied to determine the four sets of unknown expansion co-efficients a_ν, b_μ, c_n, and d_n. For the θ component of (22b) there results

$$\sum_{n=1}^{\infty}\left[\gamma_n j_n(kb)\,\frac{P_n^1(\cos\theta)}{\sin\theta} + \Gamma_n\, j_n'(kb)\,\frac{dP_n^1}{d\theta}\right]\cancel{\cos}\,\phi$$

$$+\sum_{n=1}^{\infty}\left[c_n h_n(kb)\,\frac{P_n^1(\cos\theta)}{\sin\theta} + d_n\, h_n'(kb)\,\frac{dP_n^1}{d\theta}\right]\cancel{\cos}\,\phi$$

$$=\begin{cases}\left[\sum_{\nu} a_\nu j_\nu(kb)\,\dfrac{P_\nu^1(\cos\theta)}{\sin\theta} + \sum_{\mu} b_\mu\, j_\mu'(kb)\,\dfrac{dP_\mu^1}{d\theta}\right]\cancel{\cos}\,\phi, & 0 \le \theta < \theta_0, \\ 0\,, & \theta_0 \le \theta \le \pi\,,\end{cases} \tag{27}$$

and for the ϕ component,

$$-\sum_{n=1}^{\infty}\left[\gamma_n j_n(kb)\,\frac{dP_n^1}{d\theta} + \Gamma_n\, j_n'(kb)\,\frac{P_n^1(\cos\theta)}{\sin\theta}\right]\cancel{\sin}\,\phi$$

$$-\sum_{n=1}^{\infty}\left[c_n h_n(kb)\,\frac{dP_n^1}{d\theta} + d_n\, h_n'(kb)\,\frac{P_n^1(\cos\theta)}{\sin\theta}\right]\cancel{\sin}\,\phi$$

$$=\begin{cases}-\left[\sum_{\nu} a_\nu j_\nu(kb)\,\dfrac{dP_\nu^1}{d\theta} + \sum_{\mu} b_\mu\, j_\mu'(kb)\,\dfrac{P_\mu^1(\cos\theta)}{\sin\theta}\right]\cancel{\sin}\,\phi, & 0 \le \theta < \theta_0. \\ 0\,, & \theta_0 \le \theta \le \pi\,.\end{cases} \tag{28}$$

Similiarly, for the θ-component of (22c) there results

$$\frac{1}{\pi}\sum_{n=1}^{\infty}\left[\gamma_n j_n'(kb)\,\frac{dP_n^1}{d\theta}-\Gamma_n j_n(kb)\,\frac{P_n^1(\cos\theta)}{\sin\theta}\right]\cancel{\sin}\,\phi$$

$$+\frac{1}{\pi}\sum_{n=1}^{\infty}\left[c_n h_n'(kb)\,\frac{dP_n^1}{d\theta}-d_n h_n(kb)\,\frac{P_n^1(\cos\theta)}{\sin\theta}\right]\cancel{\sin}\,\phi$$

$$=\frac{1}{\pi}\left[\sum_{\nu} a_\nu j_\nu'(kb)\,\frac{dP_\nu^1}{d\theta}-\sum_{\mu} b_\mu j_\mu(kb)\,\frac{P_\mu^1(\cos\theta)}{\sin\theta}\right]\cancel{\sin}\,\phi,\quad 0\le\theta\le\theta_o\,, \tag{29}$$

and for the ϕ-component,

$$\frac{1}{\pi}\sum_{n=1}^{\infty}\left[\gamma_n j_n'(kb)\,\frac{P_n^1(\cos\theta)}{\sin\theta}-\Gamma_n j_n(kb)\,\frac{dP_n^1}{d\theta}\right]\cancel{\cos}\,\phi$$

$$+\frac{1}{\pi}\sum_{n=1}^{\infty}\left[c_n h_n'(kb)\,\frac{P_n^1(\cos\theta)}{\sin\theta}-d_n h_n(kb)\,\frac{dP_n^1}{d\theta}\right]\cancel{\cos}\,\phi$$

$$=\frac{1}{\pi}\left[\sum_{\nu} a_\nu j_\nu'(kb)\,\frac{P_\nu^1(\cos\theta)}{\sin\theta}-\sum_{\mu} b_\mu j_\mu(kb)\,\frac{dP_\mu^1}{d\theta}\right]\cancel{\cos}\,\phi,\quad 0\le\theta\le\theta_o\,. \tag{30}$$

These four equations, (27), (28), (29), and (30), are functions of θ, (27) and (28) over the interval $0 \le \theta \le \pi$ and (29) and (30) over the interval $0 \le \theta \le \theta_o$. In the solution of Rogers and Schultz (1960) these four equations were manipulated in a process that involved differentiation with respect to θ. It is well known that an infinite series can be integrated term-by-term with non-stringent requirements on the nature of convergence, whereas term-by-term differentiation of

an infinite series is valid only with strict requirements on the convergence of the series. Since the exact nature of the convergence of the infinite series expansions in (27) through (30) is unknown, we here use an integration process, in order to avoid the problems encountered with differentiation.

First we multiply (27) by $P_m^1(\cos\theta)$, multiply (28) by $\sin\theta \frac{dP_m^1}{d\theta}$, and subtract the two results. We then integrate the resulting equation with respect to θ over the interval 0 to π. It is necessary to evaluate two integrals with limits of 0 to π and two integrals with limits 0 to θ_o. The integrals are common to boundary value problems of this type and can be evaluated by using the associated Legendre differential equation, and (24) and (26). The integral that appears as a factor in the c_n summation fortunately involves the Kronecker delta, δ_{mn}, enabling the coefficient c_m to be separated.

The coefficient d_m is separated in exactly the same manner except that (27) is multiplied by $\sin\theta \frac{dP_m^1}{d\theta}$ and (28) by $P_m^1(\cos\theta)$.

To separate a_α, (29) is multiplied by $\sin\theta \frac{dP_\alpha^1}{d\theta}$ and (30) by P_α^1 and the results added. The subscript α denotes a particular value of the infinite set ν. This equation is then integrated with respect to θ over the interval 0 to θ_o. Again the integrals can be evaluated by using the associated Legendre differential equation, and (24) and (26). Here the integral associated with the a_ν summation involves the Kronecker delta, $\delta_{\nu\alpha}$, enabling the coefficient a_α to be separated.

The coefficient b_β is separated in the same manner as is a_α, except that (29) is multiplied by P^1_β and (30) by $\sin\theta \dfrac{dP^1_\beta}{d\theta}$. The subscript β denotes a particular value of the infinite set μ. If the values of γ_n and Γ_n given by (8) are then substituted in the four separated equations, there result:

$$c_m = \frac{-i^m\,(2m+1)\;j_m(kb)}{m(m+1)\;h_m(kb)} + \frac{(2m+1)\;\sin\theta_o}{2\left[m(m+1)\right]^2\;h_m(kb)}\;\frac{dP^1_m}{d\theta}\bigg|_{\theta=\theta_o}\;\sum_{\nu}\frac{a_\nu\;\nu(\nu+1)j_\nu(kb)P^1_\nu(\cos\theta_o)}{\nu(\nu+1)-m(m+1)} \tag{31}$$

$$d_m = \frac{i^{m+1}\,(2m+1)\;j'_m(kb)}{m(m+1)\;h'_m(kb)} + \frac{(2m+1)\;P^1_m(\cos\theta_o)}{2\left[m(m+1)\right]^2\;h'_m(kb)}\;\sum_{\nu} a_\nu j_\nu(kb)P^1_\nu(\cos\theta_o)$$

$$+\;\frac{(2m+1)\;\sin\theta_o\;P^1_m(\cos\theta_o)}{2m(m+1)\;h'_m(kb)}\;\sum_{\mu}\frac{b_\mu j'_\mu(kb)}{m(m+1)-\mu(\mu+1)}\;\frac{dP^1_\mu}{d\theta}\bigg|_{\theta=\theta_o} \tag{32}$$

$$a_\alpha = \frac{\sin\theta_o\;P^1_\alpha(\cos\theta_o)}{B_\alpha j'_\alpha(kb)}\;\sum_{n=1}^{\infty}\left\{\left[c_n h'_n(kb) + \frac{i^n\,(2n+1)j'_n(kb)}{n(n+1)}\right]\frac{\frac{dP^1_n}{d\theta}\Big|_{\theta=\theta_o}}{\alpha(\alpha+1)-n(n+1)}\right\}$$

$$-\;\frac{P^1_\alpha(\cos\theta_o)}{\alpha(\alpha+1)\;\beta_\alpha j'_\alpha(kb)}\;\sum_{n=1}^{\infty}\left\{\left[d_n h_n(kb) - \frac{i^{n+1}\,(2n+1)}{n(n+1)}\;j_n(kb)\right]P^1_n(\cos\theta_o)\right\} \tag{33}$$

$$b_\beta = \frac{\sin\theta_o \left.\frac{dP^1_\beta}{d\theta}\right|_{\theta=\theta_o}}{\beta(\beta+1)\, B_\beta j_\beta(kb)} \sum_{n=1}^{\infty} \left\{ \left[d_n h_n(kb) - \frac{i^{n+1}\,(2n+1)}{n(n+1)}\, j_n(kb) \right] \frac{n(n+1) P^1_n(\cos\theta_o)}{n(n+1) - \mu(\mu+1)} \right. \tag{34}$$

where the quantities B_α and B_β are defined by

$$B_\tau = \int_0^{\theta_o} \sin\theta\, (P^1_\tau)^2\, d\theta\ . \tag{35}$$

The reader may wish to refer to Appendix A for the analytic details of the derivation of (31) thru (35).

Equations (31) through (34) could be manipulated into four equations with each set of coefficients appearing in only one equation, but the form of the end result would be less convenient for numerical computation. Therefore, (31) through (34), together with (16) through (21), (24), (26), and (35) represent the formal solution of the problem.

We have completed the solution without the necessity of satisfying boundary condition (22d), the finite energy condition at the edge of the cone. Rogers and Schultz (1960) used numerical results to show that this finite energy condition appears to be satisfied at the edge of the cone.

One of the primary objectives of the solution of this problem is to investigate the radar cross-section of the cone. The radar cross-section, σ, is defined to be

$$\sigma = \lim_{r\to\infty} 4\pi r^2 \left| \frac{\vec{S}^{s}_{II}}{\vec{S}^{i}_{II}} \right|\ , \tag{36}$$

where $\vec{S} = \frac{1}{2}\ \mathrm{Re}\left\{ \vec{E} \times \vec{H}^* \right\}$, the average Poynting vector. For our co-ordinate system, the radar cross-section evaluated at $\theta = 0$ is more

precisely termed the back scattering radar cross-section, σ_{BS}. By using some simple algebra, σ_{BS} can be shown to be expressed by

$$\sigma_{BS} = \frac{\lambda^2}{4\pi} \left| \sum_n i^n \, n(n+1) \, (c_n - id_n) \right|^2 , \tag{37}$$

where λ is the wavelength of the incident plane wave. In order to determine the back-scattering radar cross-section, then, we must first determine the sets of c_n and d_n.

7. The Numerical Solution

Equations (31) through (34) represent an infinite number of equations in an infinite number of unknown expansion coefficients. The expansion coefficients, therefore, do not enjoy the property of finality. It is important, then, to calculate as many of the coefficients as possible in order to insure that the values of the lowest order coefficients are reasonably accurate. The number of coefficients calculated in each set is designated by n_o. All numerical work was done for $\theta_o = 165^\circ$ (a cone apex angle of 30°). The calculations have been carried out for a rather large number of values of ka in order to determine rather well the details of the graph of σ_{BS} vs. ka, a being the radius of the base of the cone.

An examination of (31) through (34) indicates that the following sets of constants need to be determined: μ, ν, B_μ, B_ν, $P_n^1(\cos 165^\circ)$, $P_\nu^1(\cos 165^\circ)$, $\left.\frac{dP_n^1}{d\theta}\right|_{\theta=165^\circ}$, $\left.\frac{dP_\mu^1}{d\theta}\right|_{\theta=165^\circ}$, $j_n(kb)$, $j_\mu(kb)$, $j_\nu(kb)$,

$j_n'(kb)$, $j_\mu'(kb)$, $j_\nu'(kb)$, $h_n(kb)$, and $h_n'(kb)$. The first thirty values, each, of μ and ν, as determined from (24) and (26), were taken from Waterman's paper (1963), and these have seven-place precision. With the exception of the radial functions, the remaining sets of constants were calculated by Schultz, Bolle, and Schindler (1963), using a Burroughs Datatron 205 computer. The reader may wish to refer to their work for a detailed presentation of the methods used in calculating these constants. Appendix B herein lists the first fifty-one values in each set of these constants along with the given sets μ and ν. The radial function constants were calculated by using the infinite series representations for the spherical Bessel functions.

When these constants are substituted into (31) through (34), $4n_o$ equations in $4n_o$ unknowns result. These $4n_o$ equations must then be solved for the four sets of n_o expansion coefficients. Two different methods were used to accomplish this. First, an iterative method was used for values of ka in the Rayleigh region. Secondly, a more complicated method, but one that is usable for any value of ka (a standard matrix solution), was used for the higher values of ka.

For the iterative solution we first assume initial values of the c_n and d_n in (33) and (34) and obtain initial values of the a_ν and b_μ. These values of a_ν and b_μ are then substituted into (31) and (32) and new values for the c_n and d_n are obtained. The process is then repeated, and continued until the values of the coefficients approach a final value. This method did not converge for values of ka greater than 0.518.

The second method is more complicated but more useful since it is applicable for higher values of ka. This method involves a straightforward matrix multiplication. If n_o coefficients are to be calculated, (31) through (34) can be written in the form

$$\vec{c} = E_1 + E_2 \vec{a} \tag{38}$$

$$\vec{d} = F_1 + F_2 \vec{a} + F_3 \vec{b} \tag{39}$$

$$\vec{a} = G_1 \vec{c} + G_2 + G_3 \vec{d} + G_4 \tag{40}$$

$$\vec{b} = H_1 \vec{d} + H_2 \quad , \tag{41}$$

where

$$\vec{c} = \begin{bmatrix} c_1 \\ c_2 \\ . \\ . \\ . \\ c_{n_o} \end{bmatrix} \qquad \vec{d} = \begin{bmatrix} d_1 \\ d_2 \\ . \\ . \\ . \\ d_{n_o} \end{bmatrix} \tag{42}$$

$$\vec{a} = \begin{bmatrix} a_{\nu_1} \\ a_{\nu_2} \\ . \\ . \\ . \\ a_{\nu_{n_o}} \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} b_{\mu_1} \\ b_{\mu_2} \\ . \\ . \\ . \\ b_{\mu_{n_o}} \end{bmatrix} \tag{43}$$

E_1, F_1, G_2, G_4, and H_2 are n_o-by-1 matrices, and E_2, F_2, F_3, G_1, G_3, and H_1 are n_o-by-n_o matrices. Since only $\vec{c}$ and $\vec{d}$ are needed to calculate σ_{BS}, we substitute (40) and (41) into (38) and (39) to eliminate $\vec{a}$ and $\vec{b}$. The two resulting relations can then be written in the form

$$\left[I - E_2\, G_1\right] \vec{c} + \left[- E_2\, G_3\right] \vec{d} = \left[E_1 + E_2\, (G_2 + G_4)\right] \tag{44}$$

$$\left[- F_2\, G_1\right] \vec{c} + \left[I - F_2\, G_3 - F_3\, H_1\right] \vec{d} = \left[F_1 + F_2\, (G_2 + G_4) + F_3\, H_2\right], \tag{45}$$

where I is the identity matrix. If we define

$$\vec{x} = \begin{bmatrix} \vec{c} \\ \\ \vec{d} \end{bmatrix} , \tag{46}$$

then (44) and (45) can be expressed as

$$A\, \vec{x} = \vec{b} \quad , \tag{47}$$

and so the desired solution is

$$\vec{x} = A^{-1}\, \vec{b} \quad . \tag{48}$$

We then have the values of c_1, c_2, ... c_{n_o}, d_1, d_2, ... d_{n_o}, enabling σ_{BS} to be calculated by using (37).

All calculations were accomplished by using an IBM-7090 digital computer programmed in FORTRAN. The quantities kb and n_o were input parameters which could be changed at will. The sets of constants μ, ν, B_μ, B_ν, and the values of the associated Legendre functions were read into the machine as input data, whereas the values of the radial

functions were calculated at the beginning of the program, since the latter are dependent upon kb. In the case of the iterative method of solution, the values of a_ν, b_μ, c_n, and d_n were printed out, either after every iteration or after every fifth iteration, depending upon the speed of convergence to the final values. A subroutine for σ_{BS} was included at the end of the program, and the value of σ_{BS} was also printed. In the case of the matrix method, only the values of c_n, d_n, and σ_{BS} were printed, as the a_ν and b_μ were not computed in this latter method. Also, in the matrix method an additional input parameter s_o was used, s_o being the number of terms retained in the summations in the elements of the matrices G_2, G_4, and H_2 (see Appendix C).

8. The Numerical Results

Table 1 lists 60 calculated values of $\sigma_{BS}/\pi a^2$ for 50 different values of ka. For some values of ka, $\sigma_{BS}/\pi a^2$ was calculated for several values of n_o with ka fixed, in order to determine the sensitivity of $\sigma_{BS}/\pi a^2$ to n_o, n_o being the number of expansion coefficients calculated in each set. Nine values of $\sigma_{BS}/\pi a^2$ were calculated by using the iteration method. As mentioned previously, it was found that the iterations would not converge for values of ka above 0.518. The matrix method was then used for all calculations for ka > 0.518.

Two different programs were written using the matrix method. A maximum of only 30 coefficients (n_o = 30) could be calculated by using the first program, and the maximum value of s_o was also limited to 30.

TABLE 1

Calculated Values of the Normalized Back-Scattering Radar Cross-Section, $\sigma_{BS}/\pi a^2$

$kb^{(1)}$	$ka^{(2)}$	$n_o^{(3)}$	Iteration method		Matrix method	
			$\sigma_{BS}/\pi a^2$	number of iterations	$\sigma_{BS}/\pi a^2$	$s_o^{(4)}$
0.10	0.0259	5	3.362×10^{-6}	50		
0.10	0.0259	10	3.956×10^{-6}	50		
0.10	0.0259	15	4.035×10^{-6}	50		
0.10	0.0259	15			3.931×10^{-6}	15
0.20	0.0518	20	6.395×10^{-5}	50		
0.20	0.0518	20			6.411×10^{-5}	20
0.50	0.130	10	2.398×10^{-3}	50		
0.50	0.130	20	2.439×10^{-3}	50		
0.50	0.130	20			2.447×10^{-3}	20
1.00	0.259	25	3.618×10^{-2}	125		
1.0C	0.259	25			3.614×10^{-2}	25
1.50	0.389	25			1.591×10^{-1}	25
2.00	0.518	25	0.4132	50		
2.00	0.518	30	0.4153	85		

(1) b is the slant height of the cone

(2) a is the base radius of the cone

(3) n_o is the number of expansion coefficients calculated in each of the sets a_ν, b_μ, c_n, and d_n

(4) s_o is the number of terms retained in the summations in the elements of the matrices G_2, G_4, and H_2

$kb^{(1)}$	$ka^{(2)}$	$n_o^{(3)}$	Iteration method		Matrix method	
			$\sigma_{BS}/\pi a^2$	number of iterations	$\sigma_{BS}/\pi a^2$	$s_o^{(4)}$
2.00	0.518	30			0.4130	30
2.25	0.583	30			0.5814	30
2.50	0.648	30			0.7716	30
2.75	0.713	30			0.9723	30
3.00	0.777	30			1.164	30
3.30	0.855	30			1.363	30
3.60	0.933	30			1.530	30
3.60	0.933	33			1.537	36
4.00	1.04	30			1.734	30
4.50	1.17	30			2.050	30
5.00	1.30	30			2.371	30
5.40	1.40	30			2.476	30
6.00	1.55	30			2.405	30
6.40	1.66	30			2.297	30
6.70	1.74	30			2.152	30
7.17	1.86	45			1.841	50
7.50	1.94	45			1.577	50
7.75	2.01	45			1.460	50
8.00	2.07	45			1.412	50
8.50	2.20	45			1.462	50
9.00	2.33	45			1.495	50
9.50	2.46	45			1.367	50
10.00	2.59	45			1.077	50
10.50	2.72	45			0.8452	50
11.00	2.85	45			0.8243	50
11.50	2.98	45			0.7746	50
12.00	3.11	45			0.5292	50
12.25	3.17	45			0.4043	50
12.50	3.24	45			0.3273	50
13.00	3.36	45			0.3636	50
13.25	3.43	45			0.4176	50
13.50	3.50	45			0.4491	50
13.75	3.56	45			0.4505	50
14.00	3.62	45			0.4490	50
14.25	3.69	45			0.5056	50
14.50	3.76	45			0.6738	50
15.00	3.88	45			1.178	50
15.30	3.96	45			1.304	50
15.50	4.01	45			1.181	50
16.00	4.14	45			0.7099	50
16.50	4.27	45			0.8850	50
17.00	4.40	45			1.273	50
17.50	4.53	45			0.8129	50
18.00	4.66	45			0.7369	50
19.00	4.92	45			1.495	50
20.00	5.18	45			1.664	50

The second program offered greater flexibility, the maximum values of n_o and s_o bring 45 and 50, respectively. These latter maximum values of n_o and s_o were limited by the memory capacity of the IBM-7090 digital computer. Both of the matrix programs were used to calculate $\sigma_{BS}/\pi a^2$ for ka = 0.933, and the values obtained were the same, 1.530, for the same number of terms, n_o. This was reassuring, considering the completely dissimiliar nature of the sequence of calculations in the two matrix programs.

Fig. 3 shows a graph of $\sigma_{BS}/\pi a^2$ vs. ka, where a is the radius of the base of the cone ($a = b \sin 15^o$). For those values of $\sigma_{BS}/\pi a^2$ calculated for several different values of n_o with ka fixed, the value of $\sigma_{BS}/\pi a^2$ corresponding to the largest n_o is used in Fig. 3.

The values of $\sigma_{BS}/\pi a^2$ in the Rayleigh region ($ka < 0.4$) are not shown in Fig. 3, but it is important to note that $\sigma_{BS}/\pi a^2$ obeys extremely well the λ^{-4} law predicted for this region. Furthermore, by using an approximate method of Siegel (Siegel, 1959), applicable in the Rayleigh region, it can be shown that at ka = 0.0259 the normalized back-scattering radar cross-section is approximately 3.56×10^{-6}. The value for ka = 0.0259 from Table 1 is 3.93×10^{-6}, which agrees with the approximate value of Siegel rather well.

For higher values of ka the graph shows unexpectedly rapid fluctuations. It is believed that these are caused by convergence difficulties, especially since a curve obtained by using $n_o = 30$ instead of $n_o = 45$ showed even wilder fluctuations for $ka > 3.2$.

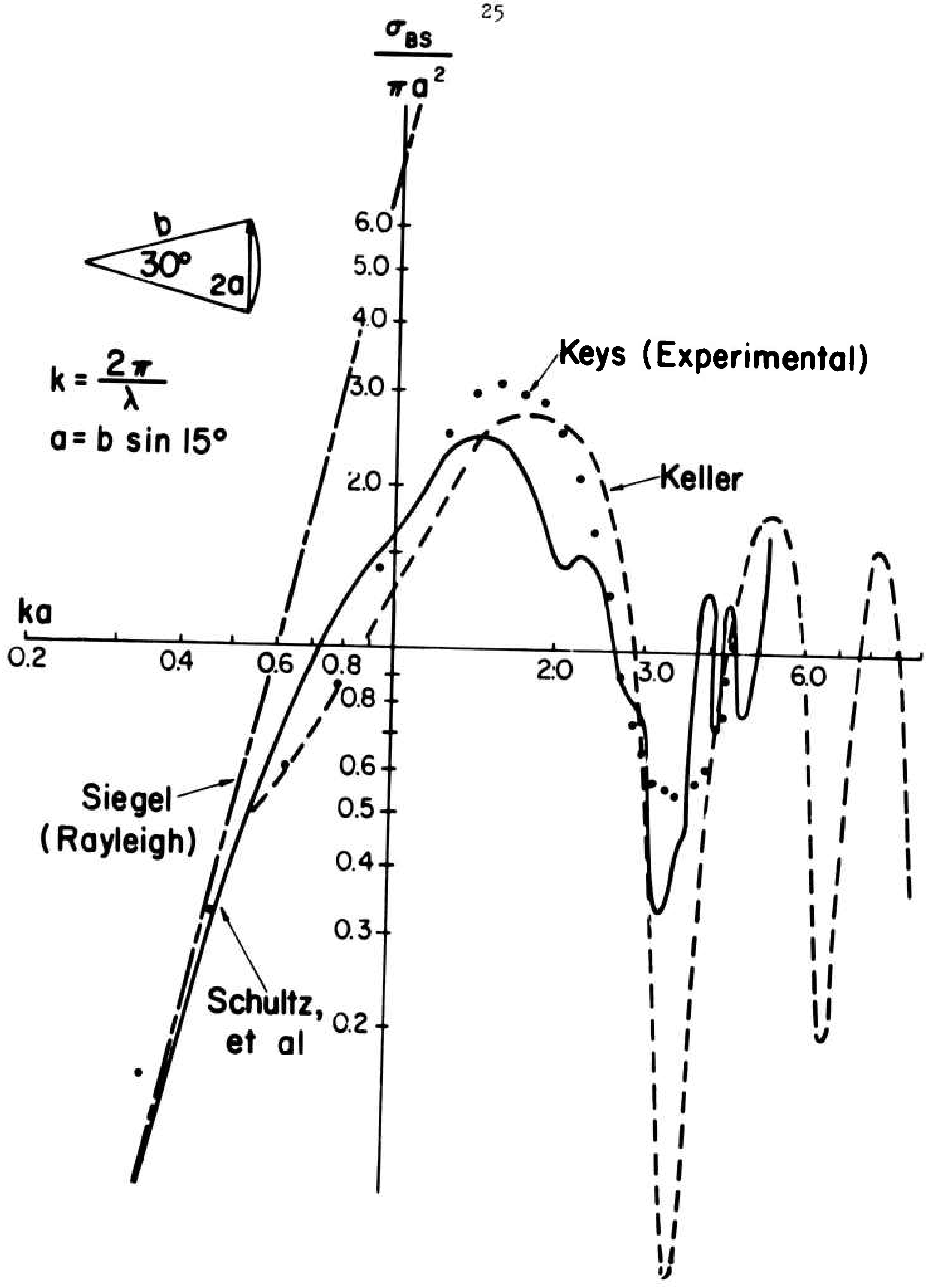

Fig. 3. Values of the Normalized Back-Scattering Radar Cross Section ($\sigma_{BS}/\pi a^2$) for a 30-Degree Perfectly Conducting Cone of Slant Height b.

Also shown in Fig. 3 are values of $\sigma_{BS}/\pi a^2$ calculated by using Keller's modified geometrical optics theory (Keller, 1960). Double diffraction effects are included.

Mr. John E. Keys of the Defence Research Telecommunications Establishment, Ottawa, Ontario, very kindly supplied the present authors with detailed data from measurements similar to those on which he and R. I. Primich reported in the Canadian Journal of Physics (1959). Mr. Keys has given his permission for the inclusion of these measurements in the present report, and they are plotted in Fig. 3. These measurements were made on flat-based cones but Mr. Keys has informed the present authors that he has made measurements on spherically-capped cones, of the type analyzed in the present work, and these measurements are indistinguishable from those made on flat-based cones.

Likewise included in Fig. 3 is the straight-line graph representing the results obtained by using Siegel's modified Rayleigh theory (Siegel, 1959).

The conclusions to be drawn from Fig. 3 are rather obvious and will not be discussed. It is of interest, however, to point out that the irregularities appearing at ka values of about 1.0, 2.2, and 2.8 in the curve illustrating the present work, occur in a region where the calculated results are believed to be accurate, so it is considered that these are bona fide irregularities. It was thought that they might be caused by diffraction from the tip of the cone, but, when this effect was included in the Keller-theory calculations, the changes in the σ-values were too minute to be noticeable.

In order to look into possible resonance effects as the cause of these irregularities, the following table was made up.

ka	1.0	2.2	2.8
kb	3.86	8.49	10.82
$\frac{2a}{\lambda}$	0.318	0.700	0.892
$\frac{b}{\lambda}$	0.614	1.352	1.723
$\frac{\pi a}{\lambda}$	0.500	1.10	1.40
$\frac{b+a}{\lambda}$	0.773	1.702	2.17

The values of $\pi a/\lambda$ make it appear that these irregularities may be resonances in response caused by current paths from top to bottom of the cone along the edge of the base. In view of the fact that the base edge of the cone is very important in determining the scattering characteristics, it is not surprising that these resonance effects occur.

ACKNOWLEDGMENTS

One of the authors (F.V.S.) wishes to acknowledge the helpfulness of discussions with Dr. R. E. Kleinman and Dr. T. B. A. Senior of the Radiation Laboratory of the University of Michigan. The assistance of Dr. I. Marx and Mr. M. R. Halsey of Purdue University in the early phases of the work was also very valuable.

In addition, Dr. C. C. Rogers, now of Rose Polytechnic Institute, and Dr. J. K. Schindler, presently at the Air Force Cambridge Research Laboratories, made major contributions which are indispensable parts of the present report.

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APPENDIX A

ANALYTIC SOLUTION FOR EXPANSION COEFFICIENTS

The solution for the expansion coefficients will now be presented in its analytic detail.

First we multiply (27) by $P^1_m(\cos\theta)$, multiply (28) by $\sin\theta \dfrac{dP^1_m}{d\theta}$, and subtract the two results, obtaining,

$$
\begin{aligned}
&\sum_{n=1}^{\infty} \gamma_n j_n(kb)\left[\frac{P^1_m P^1_n}{\sin\theta} + \sin\theta \frac{dP^1_m}{d\theta}\frac{dP^1_n}{d\theta}\right] \\
&\quad + \sum_{n=1}^{\infty} \Gamma_n j_n'(kb)\left[P^1_m \frac{dP^1_n}{d\theta} + \frac{dP^1_m}{d\theta} P^1_n\right] \\
&\quad + \sum_{n=1}^{\infty} c_n h_n(kb)\left[\frac{P^1_m P^1_n}{\sin\theta} + \sin\theta \frac{dP^1_m}{d\theta}\frac{dP^1_n}{d\theta}\right] \\
&\quad + \sum_{n=1}^{\infty} d_n h_n'(kb)\left[P^1_m \frac{dP^1_n}{d\theta} + \frac{dP^1_m}{d\theta} P^1_n\right] \\
&= \begin{cases}
\displaystyle\sum_{\nu} a_\nu j_\nu(kb)\left[\frac{P^1_m P^1_\nu}{\sin\theta} + \sin\theta \frac{dP^1_m}{d\theta}\frac{dP^1_\nu}{d\theta}\right] \\[2ex]
\quad \displaystyle + \sum_{\mu} b_\mu j_\mu'(kb)\left[P^1_m \frac{dP^1_\mu}{d\theta} + \frac{dP^1_m}{d\theta} P^1_\mu\right], \quad 0 \leq \theta < \theta_o . \\[2ex]
0, \quad \theta_o \leq \theta \leq \pi .
\end{cases}
\end{aligned}
\tag{1-1}
$$

By integrating both sides of (1-1) with respect to θ over the interval 0 to π and combining terms, we obtain,

$$\sum_{n=1}^{\infty}\left[\gamma_n j_n(kb) + c_n h_n(kb)\right]\int_0^{\pi}\left[\frac{P_m^1 P_n^1}{\sin\theta} + \sin\theta\,\frac{dP_m^1}{d\theta}\,\frac{dP_n^1}{d\theta}\right]d\theta$$

$$+\sum_{n=1}^{\infty}\left[\Gamma_n j_n'(kb) + d_n h_n'(kb)\right]\int_0^{\pi}\left[P_m^1\,\frac{dP_n^1}{d\theta} + \frac{dP_m^1}{d\theta}\,P_n^1\right]d\theta$$

$$=\sum_{\nu} a_\nu j_\nu(kb)\int_0^{\theta_0}\left[\frac{P_m^1 P_\nu^1}{\sin\theta} + \sin\theta\,\frac{dP_m^1}{d\theta}\,\frac{dP_\nu^1}{d\theta}\right]d\theta$$

$$+\sum_{\mu} b_\mu j_\mu'(kb)\int_0^{\theta_0}\left[P_m^1\,\frac{dP_\mu^1}{d\theta} + \frac{dP_m^1}{d\theta}\,P_\mu^1\right]d\theta\,. \qquad (1\text{-}2)$$

The first integral in (1-2) is a familiar integral of boundary value problems.

$$\int_0^{\pi}\left[\frac{P_m^1 P_n^1}{\sin\theta} + \sin\theta\,\frac{dP_m^1}{d\theta}\,\frac{dP_n^1}{d\theta}\right]d\theta = \frac{2\left[m(m+1)\right]^2}{2m+1}\,\delta_{mn}, \qquad (1\text{-}3)$$

where δ_{mn} is the Kronecker delta. The second and fourth integrals are easily evaluated by using $P_n^1\Big|_{\theta=0} = 0$, $P_n^1\Big|_{\theta=\pi} = 0$, and $P_\mu^1\Big|_{\theta=\theta_0} = 0$.

$$\int_0^{\pi}\left[P_m^1\,\frac{dP_n^1}{d\theta} + \frac{dP_m^1}{d\theta}\,P_n^1\right]d\theta = \int_0^{\infty} d\left[P_m^1 P_n^1\right] = \left[P_m^1 P_n^1\right]\Big|_0^{\pi} = 0 \qquad (1\text{-}4)$$

$$\int_0^{\theta_0}\left[P_m^1\,\frac{dP_\mu^1}{d\theta} + \frac{dP_m^1}{d\theta}\,P_\mu^1\right]d\theta = \int_0^{\theta_0} d\left[P_m^1 P_\mu^1\right] = \left[P_m^1 P_\mu^1\right]\Big|_0^{\theta_0} = 0 \qquad (1\text{-}5)$$

The third integral is evaluated by utilizing the associated Legendre equation,

$$\frac{d}{d\theta}\left(\sin\theta \frac{dP_n^1}{d\theta}\right) + n(n+1)\sin\theta\, P_n^1 - \frac{P_n^1}{\sin\theta} = 0 . \qquad (1\text{-}6)$$

Multiplying by P_m^1 and using the product differentiation rule, we obtain

$$\frac{P_m^1 P_n^1}{\sin\theta} + \sin\theta \frac{dP_m^1}{d\theta}\frac{dP_n^1}{d\theta} = \frac{d}{d\theta}\left(\sin\theta\, P_m^1 \frac{dP_n^1}{d\theta}\right) + n(n+1)\sin\theta\, P_m^1 P_n^1 .$$

$$(1\text{-}7)$$

Integrating from 0 to θ_o, there results

$$\int_0^{\theta_o}\left[\frac{P_m^1 P_n^1}{\sin\theta} + \sin\theta \frac{dP_m^1}{d\theta}\frac{dP_n^1}{d\theta}\right]d\theta = \left[\sin\theta\, P_m^1 \frac{dP_n^1}{d\theta}\right]\Big|_0^{\theta_o} + \int_0^{\theta_o} n(n+1)\sin\theta\, P_m^1 P_n^1\, d\theta .$$

$$(1\text{-}8)$$

The integral in (1-8) can easily be evaluated by using (1-7). If m and n are interchanged in (1-7) and the result subtracted from (1-7), the ensuing equation is

$$\left[m(m+1)-n(n+1)\right]\sin\theta\, P_m^1 P_n^1 = \frac{d}{d\theta}\left[\sin\theta\, P_m^1 \frac{dP_n^1}{d\theta} - \sin\theta\, P_n^1 \frac{dP_m^1}{d\theta}\right] .$$

$$(1\text{-}9)$$

The integral in (1-8) may now be evaluated.

$$\int_0^{\theta_o} n(n+1)\sin\theta\, P_m^1 P_n^1\, d\theta = \frac{n(n+1)}{m(m+1)-n(n+1)}\left[\sin\theta_o\left(P_m^1 \frac{dP_n^1}{d\theta} - P_n^1 \frac{dP_m^1}{d\theta}\right)\right]\Big|_0^{\theta_o}$$

$$(1\text{-}10)$$

Substituting (1-10) in (1-8) and combining terms, there results

$$\int_0^{\theta_o}\left[\frac{P_m^1 P_n^1}{\sin\theta} + \sin\theta \frac{dP_m^1}{d\theta}\frac{dP_n^1}{d\theta}\right]d\theta = \frac{m(m+1)}{m(m+1)-n(n+1)}\left[\sin\theta\, P_m^1 \frac{dP_n^1}{d\theta}\right]\Big|_0^{\theta_o}$$

$$- \frac{n(n+1)}{m(m+1)-n(n+1)}\left[\sin\theta\, P_n^1 \frac{dP_m^1}{d\theta}\right]\Big|_0^{\theta_o} . \qquad (1\text{-}11)$$

The third integral in (1-2) is then obtained by letting $n = \nu$ in (1-11) and making use of the condition (26).

$$\int_0^{\theta_0} \left[\frac{P_m^1 P_\nu^1}{\sin\theta} + \sin\theta \frac{dP_m^1}{d\theta}\frac{dP_\nu^1}{d\theta} \right] d\theta = - \frac{\nu(\nu+1)\sin\theta_0}{m(m+1)-\nu(\nu+1)} \left[P_\nu^1 \frac{dP_m^1}{d\theta} \right]\Bigg|_{\theta_0} \tag{1-12}$$

Upon substituting the integral results, (1-3), (1-4), (1-5), and (1-12), into (1-2), we obtain

$$\sum_{n=1}^{\infty} \left[\gamma_n j_n(kb) + c_n h_n(kb) \right] \frac{2\left[m(m+1)\right]^2}{2m+1} \delta_{mn}$$

$$= - \sum_{\nu} a_\nu j_\nu(kb) \frac{\nu(\nu+1)\sin\theta_0}{m(m+1)-\nu(\nu+1)} \left[P_\nu^1 \frac{dP_m^1}{d\theta} \right]\Bigg|_{\theta_0}, \tag{1-13}$$

which can be solved for c_m:

$$c_m = - \gamma_m \frac{j_m(kb)}{h_m(kb)} + \frac{(2m+1)\sin\theta_0}{2\left[m(m+1)\right]^2 h_m(kb)} \frac{dP_m^1}{d\theta}\Bigg|_{\theta=\theta_0} \sum_{\nu} \frac{a_\nu\, \nu(\nu+1) j_\nu(kb) P_\nu^1(\cos\theta_0)}{\nu(\nu+1)-m(m+1)} \tag{1-14}$$

The separation of d_m is accomplished in a similar manner by multiplying (27) by $\sin\theta \frac{dP_m^1}{d\theta}$, multiplying (28) by P_m^1, and subtracting the two results, obtaining,

$$\sum_{n=1}^{\infty} \gamma_n j_n(kb) \left[P_m^1 \frac{dP_n^1}{d\theta} + \frac{dP_m^1}{d\theta} P_n^1 \right]$$

$$+ \sum_{n=1}^{\infty} \Gamma_n j_n'(kb) \left[\frac{P_m^1 P_n^1}{\sin\theta} + \sin\theta \frac{dP_m^1}{d\theta}\frac{dP_n^1}{d\theta} \right]$$

$$+ \sum_{n=1}^{\infty} c_n h_n(kb) \left[\frac{dP_m^1}{d\theta} P_n^1 + P_m^1 \frac{dP_n^1}{d\theta} \right]$$

$$+ \sum_{n=1}^{\infty} d_n h_n'(kb) \left[\frac{P_m^1 P_n^1}{\sin\theta} + \sin\theta \frac{dP_m^1}{d\theta}\frac{dP_n^1}{d\theta} \right]$$

$$= \begin{cases} \sum_{\nu} a_\nu j_\nu(kb) \left[\frac{dP_m^1}{d\theta} P_\nu^1 + P_m^1 \frac{dP_\nu^1}{d\theta} \right] \\ \quad + \sum_{\mu} b_\mu j_\mu'(kb) \left[\frac{P_m^1 P_\mu^1}{\sin\theta} + \sin\theta \frac{dP_m^1}{d\theta}\frac{dP_\mu^1}{d\theta} \right], \quad 0 \le \theta < \theta_0 . \\ 0, \quad \theta_0 \le \theta \le \pi . \end{cases} \tag{1-15}$$

Integrating both sides of (1-15) with respect to θ over the interval 0 to π and combining terms, we obtain,

$$\sum_{n=1}^{\infty} \left[\gamma_n j_n(kb) + c_n h_n(kb) \right] \int_0^{\pi} \left[P_m^1 \frac{dP_n^1}{d\theta} + \frac{dP_m^1}{d\theta} P_n^1 \right] d\theta$$

$$+ \sum_{n=1}^{\infty} \left[\Gamma_n j_n'(kb) + d_n h_n'(kb) \right] \int_0^{\pi} \left[\frac{P_m^1 P_n^1}{\sin\theta} + \sin\theta \frac{dP_m^1}{d\theta}\frac{dP_n^1}{d\theta} \right] d\theta$$

$$= \sum_{\nu} a_\nu j_\nu(kb) \int_0^{\theta_0} \left[\frac{dP_m^1}{d\theta} P_\nu^1 + P_m^1 \frac{dP_\nu^1}{d\theta} \right] d\theta$$

$$+ \sum_{\mu} b_\mu j_\mu'(kb) \int_0^{\theta_0} \left[\frac{P_m^1 P_\mu^1}{\sin\theta} + \sin\theta \frac{dP_m^1}{d\theta}\frac{dP_\mu^1}{d\theta} \right] d\theta . \tag{1-16}$$

The values of the first and second integrals in (1-16) are given by (1-4) and (1-3), respectively. The third integral is easily evaluated.

$$\int_0^{\theta_0} \left[\frac{dP_m^1}{d\theta} P_\nu^1 + P_m^1 \frac{dP_\nu^1}{d\theta} \right] d\theta = \int_0^{\theta_0} d\left[P_m^1 P_\nu^1 \right] = P_m^1 (\cos\theta_0)\, P_\nu^1 (\cos\theta_0) \tag{1-17}$$

The fourth integral is evaluated by using (1-11) with n replaced by μ.

$$\int_0^{\theta_0} \left[\frac{P_m^1 P_\mu^1}{\sin\theta} + \sin\theta \frac{dP_m^1}{d\theta}\frac{dP_\mu^1}{d\theta} \right] d\theta = \frac{m(m+1)}{m(m+1)-\mu(\mu+1)} \sin\theta_0 \left[P_m^1 \frac{dP_\mu^1}{d\theta} \right] \Bigg|_{\theta=\theta_0} \tag{1-18}$$

By substituting these results into (1-16), we obtain

$$\sum_{n=1}^{\infty} \left[\Gamma_n j_n'(kb) + d_n h_n^1(kb)\right] \frac{2\left[m(m+1)\right]^2}{2m+1} \delta_{mn}$$

$$= \sum_{\nu} a_\nu j_\nu(kb)\, P_m^1(\cos\theta_o)\, P_\nu^1(\cos\theta_o)$$

$$+ \sum_{\mu} b_\mu j_\mu'(kb) \frac{m(m+1)\sin\theta_o}{m(m+1)-\mu(\mu+1)} \left[P_m^1 \frac{dP_\mu^1}{d\theta}\right]\Bigg|_{\theta=\theta_o}, \qquad (1\text{-}19)$$

which can be solved for d_m:

$$d_m = -\Gamma_m \frac{j_m'(kb)}{h_m'(kb)} + \frac{(2m+1)\, P_m^1(\cos\theta_o)}{2\left[m(m+1)\right]^2 h_m'(kb)} \sum_{\nu} a_\nu j_\nu(kb)\, P_\nu^1(\cos\theta_o)$$

$$+ \frac{(2m+1)\sin\theta_o\, P_m^1(\cos\theta_o)}{2m(m+1)\, h_m'(kb)} \sum_{\mu} \frac{b_\mu j_\mu'(kb)}{m(m+1)-\mu(\mu+1)} \frac{dP_\mu^1}{d\theta}\Bigg|_{\theta=\theta_o}. \qquad (1\text{-}20)$$

The separation of a_ν is accomplished by multiplying (29) by $\sin\theta \dfrac{dP_\alpha^1}{d\theta}$, multiplying (30) by P_α^1 and adding the results, obtaining,

$$\sum_{n=1}^{\infty} \gamma_n j_n'(kb) \left[\frac{P_\alpha^1 P_n^1}{\sin\theta} + \sin\theta \frac{dP_\alpha^1}{d\theta}\frac{dP_n^1}{d\theta}\right]$$

$$- \sum_{n=1}^{\infty} \Gamma_n j_n(kb) \left[P_\alpha^1 \frac{dP_n^1}{d\theta} + \frac{dP_\alpha^1}{d\theta} P_n^1\right]$$

$$+ \sum_{n=1}^{\infty} c_n h_n'(kb) \left[\frac{P_\alpha^1 P_n^1}{\sin\theta} + \sin\theta \frac{dP_\alpha^1}{d\theta}\frac{dP_n^1}{d\theta}\right]$$

$$- \sum_{n=1}^{\infty} d_n h_n(kb) \left[P_\alpha^1 \frac{dP_n^1}{d\theta} + \frac{dP_\alpha^1}{d\theta} P_n^1 \right]$$

$$= \sum_{\nu} a_\nu j_\nu'(kb) \left[\frac{P_\alpha^1 P_\nu^1}{\sin\theta} + \sin\theta \frac{dP_\alpha^1}{d\theta} \frac{dP_\nu^1}{d\theta} \right]$$

$$- \sum_{\mu} b_\mu j_\mu(kb) \left[P_\alpha^1 \frac{dP_\mu^1}{d\theta} + \frac{dP_\alpha^1}{d\theta} P_\mu^1 \right], \quad 0 \le \theta < \theta_0 . \qquad (1\text{-}21)$$

where the subscript α denotes a particular value of the infinite set of ν. Integrating both sides of (1-21) with respect to θ over the interval 0 to θ_0 and combining terms, we obtain,

$$\sum_{n=1}^{\infty} \left[\gamma_n j_n'(kb) + c_n h_n'(kb) \right] \int_0^{\theta_0} \left[\frac{P_\alpha^1 P_n^1}{\sin\theta} + \sin\theta \frac{dP_\alpha^1}{d\theta} \frac{dP_n^1}{d\theta} \right] d\theta$$

$$- \sum_{n=1}^{\infty} \left[\Gamma_n j_n(kb) + d_n h_n(kb) \right] \int_0^{\theta_0} \left[P_\alpha^1 \frac{dP_n^1}{d\theta} + \frac{dP_\alpha^1}{d\theta} P_n^1 \right] d\theta$$

$$= \sum_{\nu} a_\nu j_\nu'(kb) \int_0^{\theta_0} \left[\frac{P_\alpha^1 P_\nu^1}{\sin\theta} + \sin\theta \frac{dP_\alpha^1}{d\theta} \frac{dP_\nu^1}{d\theta} \right] d\theta$$

$$- \sum_{\mu} b_\mu j_\mu(kb) \int_0^{\theta_0} \left[P_\alpha^1 \frac{dP_\mu^1}{d\theta} + \frac{dP_\alpha^1}{d\theta} P_\mu^1 \right] d\theta . \qquad (1\text{-}22)$$

The first integral in (1-22) is evaluated by using (1-11), replacing m by α and noting that α is a particular value of the set of ν.

$$\int_0^{\theta_0} \left[\frac{P_\alpha^1 P_n^1}{\sin\theta} + \sin\theta \frac{dP_\alpha^1}{d\theta} \frac{dP_n^1}{d\theta} \right] d\theta = \frac{\alpha(\alpha+1) \sin\theta_0}{\alpha(\alpha+1) - n(n+1)} \left[P_\alpha^1 \frac{dP_n^1}{d\theta} \right] \Bigg|_{\theta=\theta_0}$$

(1-23)

The second and fourth integrals are easily evaluated.

$$\int_0^{\theta_o} \left[P_\alpha^1 \frac{dP_n^1}{d\theta} + \frac{dP_\alpha^1}{d\theta} P_n^1 \right] d\theta = \int_0^{\theta_o} d\left[P_\alpha^1 \, P_n^1 \right] = P_\alpha^1(\cos\theta_o)\, P_n^1(\cos\theta_o) \tag{1-24}$$

$$\int_0^{\theta_o} \left[P_\alpha^1 \frac{dP_\mu^1}{d\theta} + \frac{dP_\alpha^1}{d\theta} P_\mu^1 \right] d\theta = \int_0^{\theta_o} d\left[P_\alpha^1 \, P_\mu^1 \right] = \left[P_\alpha^1 \, P_\mu^1 \right] \Big|_0^{\theta_o} = 0\,. \tag{1-25}$$

The third integral is evaluated by using (1-11),

$$\int_0^{\theta_o} \left[\frac{P_\alpha^1 \, P_\nu^1}{\sin\theta} + \sin\theta \, \frac{dP_\alpha^1}{d\theta} \frac{dP_\nu^1}{d\theta} \right] d\theta = \delta_{\alpha\nu} \int_0^{\theta_o} \left[\frac{(P_\alpha^1)^2}{\sin\theta} + \sin\theta \left(\frac{dP_\alpha^1}{d\theta} \right)^2 \right] d\theta\,, \tag{1-26}$$

where $\delta_{\alpha\nu}$ is the Kronecker delta. The integral in (1-26) can be evaluated by the use of (1-7). If (1-7) is re-written with m and n replaced by α, there results

$$\frac{(P_\alpha^1)^2}{\sin\theta} + \sin\theta \left(\frac{dP_\alpha^1}{d\theta} \right)^2 = \frac{d}{d\theta}\left(\sin\theta \, P_\alpha^1 \frac{dP_\alpha^1}{d\theta} \right) + \alpha(\alpha+1) \sin\theta \, (P_\alpha^1)^2\,. \tag{1-27}$$

The integral in (1-26) is then

$$\int_0^{\theta_o} \left[\frac{(P_\alpha^1)^2}{\sin\theta} + \sin\theta \left(\frac{dP_\alpha^1}{d\theta} \right)^2 \right] d\theta = \left(\sin\theta \, P_\alpha^1 \frac{dP_\alpha^1}{d\theta} \right) \Big|_0^{\theta_o} + \alpha(\alpha+1) \int_0^{\theta_o} \sin\theta \, (P_\alpha^1)^2 \, d\theta\,. \tag{1-28}$$

The first term on the right side of (1-28) is seen to vanish, and then, if we define

$$B_\alpha = \int_0^{\theta_o} \sin\theta \, (P_\alpha^1)^2 \, d\theta\,, \tag{1-29}$$

for (1-26) there results

$$\int_0^{\theta_o} \left[\frac{P_\alpha^1 P_\nu^1}{\sin\theta} + \sin\theta \, \frac{dP_\alpha^1}{d\theta} \frac{dP_\nu^1}{d\theta} \right] d\theta = \alpha(\alpha+1) \, B_\alpha \, \delta_{\alpha\nu} \, . \qquad (1\text{-}30)$$

Upon substituting these results in (1-22), we obtain

$$\sum_{n=1}^{\infty} \left[\gamma_n j_n'(kb) + c_n h_n'(kb) \right] \frac{\alpha(\alpha+1) \sin\theta_o}{\alpha(\alpha+1) - n(n+1)} \left[P_\alpha^1 \frac{dP_n^1}{d\theta} \right] \Big|_{\theta=\theta_o}$$

$$- \sum_{n=1}^{\infty} \left[\Gamma_n j_n(kb) + d_n h_n(kb) \right] P_\alpha^1 (\cos\theta_o) \, P_n^1 (\cos\theta_o)$$

$$= \sum_{\nu} a_\nu j_\nu'(kb) \;\; \alpha(\alpha+1) \, B_\alpha \, \delta_{\alpha\nu} \, , \qquad (1\text{-}31)$$

and solving for a_α ,

$$a_\alpha = \frac{\sin\theta_o \, P_\alpha^1 (\cos\theta_o)}{B_\alpha \, j_\alpha'(kb)} \sum_{n=1}^{\infty} \frac{\gamma_n j_n'(kb) + c_n h_n'(kb)}{\alpha(\alpha+1) - n(n+1)} \frac{dP_n^1}{d\theta} \Big|_{\theta=\theta_o}$$

$$- \frac{P_\alpha^1 (\cos\theta_o)}{\alpha(\alpha+1) \, B_\alpha j_\alpha'(kb)} \sum_{n=1}^{\infty} \left[\Gamma_n j_n(kb) + d_n h_n(kb) \right] P_n^1 (\cos\theta_o) \, .$$

(1-32)

The last coefficient to be separated is b_μ. This is accomplished by multiplying (29) by P_β^1 , multiplying (30) by $\sin\theta \, \frac{dP_\beta^1}{d\theta}$, and adding the results, obtaining,

$$\sum_{n=1}^{\infty} \gamma_n j_n'(kb) \left[P_\beta^1 \frac{dP_n^1}{d\theta} + \frac{dP_\beta^1}{d\theta} P_n^1 \right]$$

$$- \sum_{n=1}^{\infty} \Gamma_n j_n(kb) \left[\frac{P_\beta^1 P_n^1}{\sin\theta} + \sin\theta \, \frac{dP_\beta^1}{d\theta} \frac{dP_n^1}{d\theta} \right]$$

$$+\sum_{n=1}^{\infty} c_n h_n'(kb) \left[P_\beta^1 \frac{dP_n^1}{d\theta} + \frac{dP_\beta^1}{d\theta} P_n^1 \right]$$

$$-\sum_{n=1}^{\infty} d_n h_n(kb) \left[\frac{P_\beta^1 P_n^1}{\sin\theta} + \sin\theta \frac{dP_\beta^1}{d\theta} \frac{dP_n^1}{d\theta} \right]$$

$$= \sum_{\nu} a_\nu j_\nu'(kb) \left[P_\beta^1 \frac{dP_\nu^1}{d\theta} + \frac{dP_\beta^1}{d\theta} P_\nu^1 \right]$$

$$-\sum_{\mu} b_\mu j_\mu(kb) \left[\frac{P_\beta^1 P_\mu^1}{\sin\theta} + \sin\theta \frac{dP_\beta^1}{d\theta} \frac{dP_\mu^1}{d\theta} \right], \quad 0 \leq \theta < \theta_o, \qquad (1\text{-}33)$$

where the subscript β denotes a particular value of the infinite set μ. Upon integrating both sides of (1-33) with respect to θ over the interval 0 to θ_o and combining terms, we obtain,

$$\sum_{n=1}^{\infty} \left[\gamma_n j_n'(kb) + c_n h_n'(kb) \right] \int_0^{\theta_o} \left[P_\beta^1 \frac{dP_n^1}{d\theta} + \frac{dP_\beta^1}{d\theta} P_n^1 \right] d\theta$$

$$-\sum_{n=1}^{\infty} \left[\Gamma_n j_n(kb) + d_n h_n(kb) \right] \int_0^{\theta_o} \left[\frac{P_\beta^1 P_n^1}{\sin\theta} + \sin\theta \frac{dP_\beta^1}{d\theta} \frac{dP_n^1}{d\theta} \right] d\theta$$

$$= \sum_{\nu} a_\nu j_\nu'(kb) \int_0^{\theta_o} \left[P_\beta^1 \frac{dP_\nu^1}{d\theta} + \frac{dP_\beta^1}{d\theta} P_\nu^1 \right] d\theta$$

$$-\sum_{\mu} b_\mu j_\mu(kb) \int_0^{\theta_o} \left[\frac{P_\beta^1 P_\mu^1}{\sin\theta} + \sin\theta \frac{dP_\beta^1}{d\theta} \frac{dP_\mu^1}{d\theta} \right] d\theta . \qquad (1\text{-}34)$$

The first integral in (1-34) is easily evaluated, noting that β is a particular value of the set μ.

$$\int_0^{\theta_0}\left[P_\beta^1\frac{dP_n^1}{d\theta}+\frac{dP_\beta^1}{d\theta}P_n^1\right]d\theta=\int_0^{\theta_0}d\left[P_\beta^1 P_n^1\right]=\left[P_\beta^1 P_n^1\right]\Big|_0^{\theta_0}=0 \tag{1-35}$$

The second integral is evaluated by using (1-11) and replacing m by β, obtaining,

$$\int_0^{\theta_0}\left[\frac{P_\beta^1 P_n^1}{\sin\theta}+\sin\frac{dP_\beta^1}{d\theta}\frac{dP_n^1}{d\theta}\right]d\theta=\frac{n(n+1)\sin\theta_0}{n(n+1)-\beta(\beta+1)}\left[P_n^1\frac{dP_\beta^1}{d\theta}\right]\Big|_{\theta=\theta_0}. \tag{1-36}$$

The third integral in (1-34) also is elementary.

$$\int_0^{\theta_0}\left[P_\beta^1\frac{dP_\nu^1}{d\theta}+\frac{dP_\beta^1}{d\theta}P_\nu^1\right]d\theta=\int_0^{\theta_0}d\left[P_\beta^1 P_\nu^1\right]d\theta=\left[P_\beta^1 P_\nu^1\right]\Big|_0^{\theta_0}=0 \tag{1-37}$$

The fourth integral is evaluated by using (1-11),

$$\int_0^{\theta_0}\left[\frac{P_\beta^1 P_\mu^1}{\sin\theta}+\sin\theta\frac{dP_\beta^1}{d\theta}\frac{dP_\mu^1}{d\theta}\right]d\theta=\delta_{\beta\mu}\int_0^{\theta_0}\left[\frac{(P_\beta^1)^2}{\sin\theta}+\sin\theta\left(\frac{dP_\beta^1}{d\theta}\right)^2\right]d\theta\ , \tag{1-38}$$

where $\delta_{\beta\mu}$ is the Kronecker delta. The integral in (1-38) is evaluated by using (1-28) with α replaced by β.

$$\int_0^{\theta_0}\left[\frac{(P_\beta^1)^2}{\sin\theta}+\sin\theta\left(\frac{dP_\beta^1}{d\theta}\right)^2\right]d\theta=\left(\sin\theta\, P_\beta^1\frac{dP_\beta^1}{d\theta}\right)\Big|_0^{\theta_0}+\beta(\beta+1)\int_0^{\theta_0}\sin\theta\,(P_\beta^1)^2\,d\theta \tag{1-39}$$

The first term on the right side of (1-39) is seen to vanish, and then, if we define

$$B_\beta=\int_0^{\theta_0}\sin\theta\,(P_\beta^1)^2\,d\theta\ , \tag{1-40}$$

for (1-38) there results

$$\int_0^{\theta_o} \left[\frac{P_\beta^1 P_\mu^1}{\sin\theta} + \sin\theta \frac{dP_\beta^1}{d\theta}\frac{dP_\mu^1}{d\theta} \right] d\theta = \beta(\beta+1)\, B_\beta\, \delta_{\beta\mu} . \tag{1-41}$$

Upon substituting the results of these integrations into (1-34), we obtain

$$-\sum_{n=1}^{\infty} \left[\Gamma_n j_n(kb) + d_n h_n(kb)\right] \frac{n(n+1)\sin\theta_o}{n(n+1)-\beta(\beta+1)} \left[P_n^1 \frac{dP_\beta^1}{d\theta} \right] \Big|_{\theta=\theta_o}$$

$$= -\sum_{\mu} b_\mu j_\mu(kb)\, \beta(\beta+1)\, B_\beta\, \delta_{\beta\mu} , \tag{1-42}$$

and solving for b_β ,

$$b_\beta = \frac{\sin\theta_o}{\beta(\beta+1)\, B_\beta j_\beta(kb)} \frac{dP_\beta^1}{d\theta}\Big|_{\theta=\theta_o} \sum_{n=1}^{\infty} \frac{\left[\Gamma_n j_n(kb) + d_n h_n(kb)\right]}{n(n+1)-\beta(\beta+1)}\, n(n+1)\, P_n^1(\cos\theta_o). \tag{1-43}$$

Equations (1-14), (1-20), (1-32), and (1-43) represent the formal solution for the expansion co-efficients, and, when the values of γ_n and Γ_n are substituted by using (8), are equivalent to (31) through (34).

APPENDIX B

LEGENDRE FUNCTION CONSTANTS

n	$\mu^{(1)(2)}$	$\nu^{(1)(3)}$	$B_\mu^{(4)}$	$B_\nu^{(4)}$
1	1.031631	0.967140	1.310	1.35806
2	2.084434	1.918899	2.346	2.42491
3	3.149929	2.887078	3.347	3.37945
4	4.223096	3.887853	4.341	4.28564
5	5.301087	4.917100	5.330	5.18033
6	6.382249	5.965629	6.323	6.09038
7	7.465580	7.026428	7.168	7.03236
8	8.550454	8.095125	8.206	8.058
9	9.63645	9.169073	9.227	9.009
10	10.72329	10.24665	10.24	9.973
11	11.81078	11.32681	11.25	10.95
12	12.89879	12.40890	12.25	11.93
13	13.98718	13.49242	13.26	12.91
14	15.07592	14.57706	14.26	13.89
15	16.16491	15.66258	15.26	14.88
16	17.25414	16.74882	16.26	15.87
17	18.34354	17.83562	17.26	16.86
18	19.43311	18.92291	18.26	17.86
19	20.52280	20.01059	19.26	18.85
20	21.61262	21.09860	20.26	19.84
21	22.70252	22.18690	21.26	20.84
22	23.79253	23.27545	22.26	21.83
23	24.88260	24.36421	23.26	22.83
24	25.97275	25.45315	24.26	23.83
25	27.06294	26.54226	25.26	24.82
26	28.15320	27.63151	26.26	25.82
27	29.24349	28.72088	27.26	26.82
28	30.33385	29.81037	28.26	27.82
29	31.42421	30.89995	29.26	28.81
30	32.51465	31.98963	30.26	29.81

(1) Donated by Dr. P. C. Waterman of AVCO.

(2) Determined from $P_\mu^1(\cos 165^\circ) = 0$.

(3) Determined from $\left.\dfrac{dP_\nu^1}{d\theta}\right|_{\theta=165^\circ} = 0$.

(4) Defined by $B_\tau = \int_0^{165^\circ} \sin\theta \, (P_\tau^1)^2 \, d\theta$.

31	33.61	33.08	31.26	30.81
32	34.70	34.17	32.26	31.81
33	35.79	35.26	33.26	32.804
34	36.88	36.35	34.26	33.803
35	37.97	37.44	35.26	34.801
36	39.06	38.53	36.26	35.800
37	40.15	39.62	37.26	36.798
38	41.24	40.71	38.26	37.797
39	42.33	41.80	39.26	38.796
40	43.42	42.89	40.26	39.794
41	44.51	43.98	41.26	40.793
42	45.60	45.07	42.26	41.792
43	46.69	46.16	43.26	42.791
44	47.78	47.25	44.26	43.790
45	48.87	48.34	45.26	44.789
46	49.96	49.43	46.26	45.788
47	51.05	50.52	47.26	46.787
48	52.14	51.61	48.26	47.787
49	53.24	52.70	49.26	48.786
50	54.33	53.79	50.26	49.785
51	55.42	54.88	51.26	50.784

n	$P_n^1(\cos 165^\circ)$	$P_\nu^1(\cos 165^\circ)$	$\frac{dP_n^1}{d\theta}\Big\vert_{\theta=165^\circ}$	$\frac{dP_\mu^1}{d\theta}\Big\vert_{\theta=165^\circ}$
1	-0.25881924	-0.52346792	0.96592579	1.886
2	0.750	1.4156256	-2.5980742	-5.177
3	-1.4228831	-2.3050079	4.3396881	9.503
4	2.2069309	2.98087	-5.4575812	-14.65
5	-3.0177961	-3.50544	5.1518426	20.48
6	3.7646396	3.92272	-2.6831313	-26.91
7	-4.3581639	-4.27745	-2.556614	32.93
8	4.748	4.603	11.19	-40.63
9	-4.797	-4.891	-22.08	48.73
10	4.514	5.159	35.26	-57.23
11	-3.884	-5.413	-49.72	66.12
12	2.922	5.655	64.10	-75.39
13	-1.675	-5.886	-76.74	85.03
14	0.2149	6.109	85.84	-95.03
15	1.364	-6.323	-89.66	105.4
16	-2.953	6.530	86.71	-116.1
17	4.433	-6.732	-75.92	127.1
18	-5.690	6.927	56.81	-138.4
19	6.618	-7.117	-29.64	158.0
20	-7.133	7.302	-4.551	-162.0
21	7.176	-7.483	43.92	174.2
22	-6.720	7.659	-85.90	-186.7
23	5.775	-7.831	127.3	199.5
24	-4.388	8.000	-164.8	-212.6
25	2.637	-8.166	194.7	226.0
26	-0.6337	8.328	-213.6	-239.6
27	-1.491	-8.487	218.7	253.5
28	3.591	8.643	-208.2	-267.6
29	-5.516	-8.796	180.8	282.0
30	7.126	8.947	-137.1	-296.6
31	-8.296	-9.096	78.73	311.5
32	8.929	9.242	-8.613	-326.6
33	-8.964	-9.385	-68.98	342.0
34	8.380	9.527	148.8	-357.5
35	-7.199	-9.666	-225.2	373.4
36	5.487	9.804	291.9	-389.4
37	-3.349	-9.940	-343.2	405.7
38	0.9210	10.07	374.0	-422.1
39	1.634	-10.21	-380.2	438.8
40	-4.142	10.34	359.5	-455.7
41	6.427	-10.46	-311.2	472.9
42	-8.324	10.59	237.0	-490.2
43	9.693	-10.72	-140.3	507.7
44	-10.42	10.84	26.61	-525.5
45	10.45	-10.96	96.96	543.4
46	-9.765	11.09	-222.1	-561.5
47	8.388	-11.21	339.7	579.9
48	-6.404	11.33	-440.9	-598.4
49	3.937	-11.44	517.1	617.1
50	-1.149	11.56	-561.4	-636.0
51	-1.773	-11.68	568.5	655.1

APPENDIX C

DEFINITIONS OF MATRIX ELEMENTS

The elements of the matrices in (38) through (41) are given by (31) through (34). If we let $[A]_{ij}$ denote the j^{th} element of the i^{th} row of a matrix A, then the matrix elements are defined by

$$\left[E_1\right]_m = \frac{-i^m (2m+1)\; j_m(kb)}{m(m+1)\; h_m(kb)} \tag{3-1}$$

$$\left[E_2\right]_{mn} = \frac{(2m+1)\sin\theta_o \left.\frac{dP_m^1}{d\theta}\right|_{\theta=\theta_o} \nu_n(\nu_n+1)\; j_{\nu_n}(kb)\; P^1_{\nu_n}(\cos\theta_o)}{2\left[m(m+1)\right]^2 h_m(kb)\left[\nu_n(\nu_n+1) - m(m+1)\right]} \tag{3-2}$$

$$\left[F_1\right]_m = \frac{i^{m+1}(2m+1)\; j_m'(kb)}{m(m+1)\; h_m'(kb)} \tag{3-3}$$

$$\left[F_2\right]_{mn} = \frac{(2m+1)\; P_m^1(\cos\theta_o)\; j_{\nu_n}(kb)\; P^1_{\nu_n}(\cos\theta_o)}{2\left[m(m+1)\right]^2 h'_m(kb)} \tag{3-4}$$

$$\left[F_3\right]_{mn} = \frac{(2m+1)\sin\theta_o\; P_m^1(\cos\theta_o)\; j'_{\mu_n}(kb) \left.\frac{dP^1_{\mu_n}}{d\theta}\right|_{\theta=\theta_o}}{2m(m+1)\; h_m'(kb)\left[m(m+1) - \mu_n(\mu_n+1)\right]} \tag{3-5}$$

$$\left[G_1\right]_{mn} = \frac{\sin\theta_o\; P^1_{\nu_m}(\cos\theta_o)\; h_n'(kb) \left.\frac{dP_n^1}{d\theta}\right|_{\theta=\theta_o}}{j'_{\nu_m}(kb)\; B_{\nu_m}\left[\nu_m(\nu_m+1) - n(n+1)\right]} \tag{3-6}$$

$$\left[G_2\right]_m = \frac{\sin\theta_o\; P^1_{\nu_m}(\cos\theta_o)}{j'_{\nu_m}(kb)\; B_{\nu_m}} \sum_{n=1}^{S_o} \frac{i^n (2n+1)\; j_n'(kb) \left.\frac{dP_n^1}{d\theta}\right|_{\theta=\theta_o}}{n(n+1)\left[\nu_m(\nu_m+1) - n(n+1)\right]} \tag{3-7}$$

$$\left[G_3\right]_{mn} = \frac{- P^1_{\nu_m}(\cos\theta_o)\, h_n(kb)\, P^1_n(\cos\theta_o)}{\nu_m(\nu_m+1)\, B_{\nu_m}\, j'_{\nu_m}(kb)} \tag{3-8}$$

$$\left[G_4\right]_m = \frac{P^1_{\nu_m}(\cos\theta_o)}{\nu_m(\nu_m+1)\, B_{\nu_m}\, j'_{\nu_m}(kb)} \sum_{n=1}^{s_o} \frac{i^{n+1}\,(2n+1)\, j_n(kb)\, P^1_n(\cos\theta_o)}{n(n+1)} \tag{3-9}$$

$$\left[H_1\right]_{mn} = \frac{\sin\theta_o \left.\frac{dP^1_{\mu_m}}{d\theta}\right|_{\theta=\theta_o} h_n(kb)\, n(n+1)\, P^1_n(\cos\theta_o)}{\mu_m(\mu_m+1)\, B_{\mu_m}\, j_{\mu_m}(kb)\left[n(n+1) - \mu_m(\mu_m+1)\right]} \tag{3-10}$$

$$\left[H_2\right]_m = \frac{- \sin\theta_o \left.\frac{dP^1_{\mu_m}}{d\theta}\right|_{\theta=\theta_o}}{\mu_m(\mu_m+1)\, B_{\mu_m}\, j_{\mu_m}(kb)} \sum_{n=1}^{s_o} \frac{i^{n+1}\,(2n+1)\, j_n(kb)\, P^1_n(\cos\theta_o)}{\left[n(n+1) - \mu_m(\mu_m+1)\right]} . \tag{3-11}$$

μ_m and ν_m represent the m^{th} value of the sets μ and ν, respectively.

School of Electrical Engineering, Purdue University, Lafayette, Indiana, THE THEORETICAL AND NUMERICAL DETERMINATION OF THE RADAR CROSS SECTION OF A FINITE CONE by F. V. Schultz, et al. Air Force Cambridge Research Laboratories, Bedford, Mass. Scientific Report – Number AFCRL–64–658, 46 pages, August, 1964.

Unclassified Report

This investigation of the radar cross section of a finite cone can be divided into three areas. First, the exact solution for the scattering of a plane electromagnetic wave by a finite cone is presented. Rigorous electromagnetic theory is used in the solution, and no approximations are made. Second, methods of obtaining numerical results for the radar cross section from the analytic solution by using a digital computer are discussed. The third area is a presentation and discussion of the numerical results obtained.

1. Backscattering from a finite cone

I. Project No. 5635
   Task No. 563502
II. Contract AF 19(628)–1691
III. F. V. Schultz, et al.
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